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1995 J. Phys. A: Math. Gen. 28 4609

(http://iopscience.iop.org/0305-4470/28/16/018)

J. Phys. A: Math. Gen. 28 (1995) 4609–4621. Printed in the UK

Wave splitting and lattice dynamics

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Received 13 February 1995

Abstract. A wave-splitting approach used elsewhere to solve a black-hole scattering problem is adapted to the formal solution of arbitrary self-adjoint wave equations in $1+1$ dimensions and yields potentially useful results in this more general case. It is then shown that the self-adjoint wave equation being solved, and the non-self-adjoint linear wave equations satisfied by the one-way component waves, are naturally related to a pair of motions of the $(1+1)$-dimensional Toda lattice that together comprise a motion of the Kac–van Moerbeke lattice. This provides a partial explanation for the peculiar fact that every motion of the Kac–van Moerbeke lattice can be viewed as two interpolated motions of the Toda lattice.

1. Introduction

In 1984 Torrence and Couch [1] applied a novel wave-splitting method to the solution of a specific scattering problem in general relativity and calculated the transmission and reflection coefficients as a function of frequency for a spherical scalar wave incident on a black hole. The results were gratifying as the scheme proved to be equally effective at all frequencies. It was evident that the method should be applicable to a wider class of problems, and in this paper we apply our wave decomposition approach to solving a large class of linear wave equations. We first obtain formal series solutions of the *entire class* of linear self-adjoint wave equations in $1+1$ dimensions; these series decompose in a natural way into one-way waves. Under some circumstances, including those holding in [1], these series can be shown to converge absolutely to rigorous solutions; however, our main result, which is of a more formal nature, is the establishment of a relationship between the wave-splitting method described here, and a well known but unintuitive fact concerning the motions of the two nonlinear dynamical lattices usually referred to in the literature as the Kac–van Moerbeke lattice (KvML) [2, 3] and the Toda lattice (TL) [4]. Loosely put, this fact is that every motion of a KvML is two interpolated TL motions. Whether our result helps to 'explain' this peculiarity about lattice motions is a matter of taste, but it establishes a connection between it and something intuitively appealing in another area of mathematical physics, i.e. scattering theory.

The contents of the paper are organized as follows. In section 2 we define formal *reflection series* representations for two particular types of basic solutions of the class of self-adjoint wave equations. The series lead in a natural way to the splitting of each of these basic solutions into two one-way component waves that can be useful in formulating scattering problems, and to the identification of a pair of linear second-order wave equations, not self-adjoint, each satisfied by a distinct pair of these four component waves. In section 3 we review the way in which the iterated application of the classical Laplace transformation [5] to a given linear wave equation in $1+1$ dimensions produces a doubly infinite sequence of equivalent linear wave equations, and the fact that the equations defining this equivalence relation are precisely those defining TL dynamics, as was pointed out in [6]. In section 4 we review the definition of KvML dynamics and the way in which a motion of the KvML is related to two motions of the TL, including an anomaly in that relationship that is not emphasized elsewhere and has a bearing on our new results. In section 5 we show that under the bijection described in section 3 the equivalence class of wave equations containing the self-adjoint equation we wish to solve, and a second equivalence class containing both of the non-self-adjoint equations satisfied by the one-way component waves, are mapped into two TL motions that are related by a KvML motion in the way described in section 4. Most of the results in sections 3 and 4 are reviews of known facts, however, those in section 2 seem to be mainly new (and may be of practical value) and the result in section 5 is apparently entirely new and unexpected. In the conclusion we compare the wave decompositions obtained in section 2 with some of the earlier work on wave-splittings [7–9] and discuss some open problems suggested by our results in section 5.

2. Wave splitting by reflection

The general linear wave equation in $1+1$ dimensions can be written in the form

$$(\partial_v k \partial_u - j)\phi = 0 \tag{2.1}$$

by appropriate transformations on the dependent and independent variables, where k and j are given functions of characteristic coordinates u and v. If (2.1) is specialized to

$$(\partial^2_{vu} - j)\phi = 0 \tag{2.2}$$

i.e. if we assume $k(u, v) = 1$, then we are dealing with the $(1+1)$-dimensional case of the class of equations known in the mathematical literature as *formally self-adjoint* equations. There is a simple and intuitively satisfying approach to the solution of (2.2) that is of interest in its own right and also leads to a natural linear decomposition of its solutions into one-way component waves. It is natural to regard a field that satisfies (2.2) as propagating along the characteristics of the equation, which are the lines of constant u and v respectively, but undergoing a process of partial reflection at every field point. As a result, one can hope that if one had, for example, characteristic data $a(v)$ at past null infinity, i.e. at $u = -\infty$, the field at a general field point coordinatized by u, v would be the sum of

$$a(v) \qquad \int_{-\infty}^{v} \mathrm{d}v'\, b(u, v')a(v') \qquad \int_{-\infty}^{u} \mathrm{d}u'\, f(u', v) \int_{-\infty}^{v} \mathrm{d}v'\, b(u', v')a(v') \qquad \ldots$$

where the first term represents incoming data that reached the field point from past null infinity without reflection, the second term represents that which reached there after one reflection, where the single integration sums over all the events where that reflection might have occurred, the third term represents data which reached the field point after two reflections, and so on. Here the functions $f(u, v)$ and $b(u, v)$ that alternate in the integrands encode the strength of the local reflections that occur at each point. If we generalize the lower limits of the two types of integrals to be the labels of an arbitrary pair of intersecting characteristics we are led to conjecture that (2.2) might be satisfied, at least formally, by each of the two *reflection series*

$$\phi^{\mathrm{A}} \equiv a(v) + \int_{v_0}^{v} \mathrm{d}v'\, ba + \int_{u_0}^{u} \mathrm{d}u'\, f \int_{v_0}^{v} \mathrm{d}v'\, ba + \cdots \tag{2.3}$$

and

$$\phi^{\mathrm{R}} \equiv r(u) + \int_{u_0}^{u} \mathrm{d}u'\, fr + \int_{v_0}^{v} \mathrm{d}v'\, b \int_{u_0}^{u} \mathrm{d}u'\, fr + \cdots \tag{2.4}$$

where $a(v)$ and $r(u)$ are independent and arbitrary data functions. If we are permitted to rearrange these series it is clear that the sum of the terms of (2.3) containing an even number of integral signs, corresponding to data that has been reflected an even number of times, might reasonably be viewed as that portion of ϕ^{A} moving along characteristics defined by a constant v, while the sum of the terms with an odd number of integrals corresponds to that part moving along characteristics defined by a constant u, and *vice versa* for (2.4). This motivates us to write

$$\phi^{\mathrm{A}} = \phi^{\mathrm{Aev}} + \phi^{\mathrm{Aod}} \qquad \phi^{\mathrm{R}} = \phi^{\mathrm{Rev}} + \phi^{\mathrm{Rod}} \tag{2.5}$$

where

$$\begin{aligned}
\phi^{\mathrm{Aev}} &\equiv a(v) + \int_{u_0}^{u} f \int_{v_0}^{v} ba + \cdots \\
\phi^{\mathrm{Aod}} &\equiv \int_{v_0}^{v} ba + \int_{v_0}^{v} b \int_{u_0}^{u} f \int_{v_0}^{v} ba + \cdots \\
\phi^{\mathrm{Rev}} &\equiv r(u) + \int_{v_0}^{v} b \int_{u_0}^{u} fr + \cdots \\
\phi^{\mathrm{Rod}} &\equiv \int_{u_0}^{u} fr + \int_{u_0}^{u} f \int_{v_0}^{v} b \int_{u_0}^{u} fr + \cdots
\end{aligned} \tag{2.6}$$

and where we have ceased writing the differentials for simplicity. It is these series, which lend themselves to a nice formulation of scattering problems, that were successfully applied in [1]. If we continue to take a relaxed attitude about interchanging limit processes it is easy to derive from (2.6) that

$$\partial_u \phi^{\mathrm{Aev}} = f(u, v)\phi^{\mathrm{Aod}} \qquad \partial_v \phi^{\mathrm{Aod}} = b(u, v)\phi^{\mathrm{Aev}} \tag{2.7}$$

and

$$\partial_u \phi^{\mathrm{Rod}} = f(u, v)\phi^{\mathrm{Rev}} \qquad \partial_v \phi^{\mathrm{Rev}} = b(u, v)\phi^{\mathrm{Rod}}. \tag{2.8}$$

It follows in a few steps from (2.7) that

$$\partial^2_{uv}\phi^{\mathrm{A}} = j\phi^{\mathrm{A}} \qquad \text{if and only if } \partial_v f + fb = j(u, v) \tag{2.9}$$

and similarly from (2.8) that

$$\partial^2_{uv}\phi^{\mathrm{R}} = j\phi^{\mathrm{R}} \qquad \text{if and only if } \partial_u b + bf = j(u, v) \tag{2.10}$$

so we have shown the following theorem.

Theorem 1. Equations (2.5) with the definitions (2.6) yield formal solutions of (2.2) if and only if the two functions $f(u, v)$ and $b(u, v)$ satisfy the coupled first-order nonlinear partial differential equations

$$\partial_v f + fb = j \qquad \partial_u b + bf = j. \tag{2.11}$$

It might appear that theorem 1 is of little use since we have merely replaced the task of solving one second-order partial differential equation by that of solving two coupled first-order ones, but this is not the case. If we are given any function $\pi(u, v)$ and define

$$b \equiv (\partial_v \pi)/\pi \qquad f \equiv (\partial_u \pi)/\pi \tag{2.12}$$

it follows immediately that

$$\partial^2_{uv}\pi = (\partial_v f + fb)\pi = (\partial_u b + bf)\pi \tag{2.13}$$

and we have a second theorem.

Theorem 2. The functions $f(u, v)$ and $b(u, v)$ defined by (2.12) satisfy our basic equations (2.11) if and only if $\pi(u, v)$ is a particular solution of (2.2).

When this result is combined with theorem 1 it can be seen that each *particular* solution of (2.2) yields an intuitively satisfying formal representation of the *general* solution, as well as a particular local break-up of that solution into the one-way waves given by (2.6). It is easy to see that there are choices of $f(u, v)$ and $b(u, v)$, $a(v)$ and $r(u)$, and v_0 and u_0, for which (2.3) and (2.4) are not well-defined; however, as was illustrated in [1] there are also realistic cases where they converge to rigorous and useful solutions of (2.2).

A pair of formulae that are essential in obtaining our result in section 5 follow immediately from (2.7) and (2.8), as the latter imply that

$$\left(\partial_v \frac{1}{f}\partial_u - b\right)\phi = 0 \tag{2.14}$$

where ϕ can be either ϕ^{Aev} or ϕ^{Rod}, and that

$$\left(\partial_u \frac{1}{b}\partial_v - f\right)\psi = 0 \tag{2.15}$$

where ψ can be either ϕ^{Aod} or ϕ^{Rev}. The fact that each of the one-way components used to construct ϕ^{A} and ϕ^{R} satisfies one of the two non-self-adjoint linear wave equations (2.14) or (2.15) is interesting in its own right, and is central to the results in section 5. The logical relationship between the self-adjoint equation that we wish to solve, (2.2), and the pair of equations that govern the one-way waves, (2.14) and (2.15), is worth an explicit comment. On the one hand if we begin with (2.2), i.e. with $j(u, v)$, we can generate an infinite set of pairs f and b, one pair for each particular solution of (2.2), and thus an infinte set of pairs of non-self-adjoint wave equations, (2.14) and (2.15), satisfied by the corresponding one-way waves. On the other hand if we start with f and b, they generate a unique $j(u, v)$ through (2.11) and, thus, a unique equation (2.2) if and only if they satisfy the compatibility condition

$$\partial_v f = \partial_u b. \tag{2.16}$$

In what follows we shall be exclusively interested in two special classes of wave equations: (i) the family of self-adjoint equations which are characterized by $j(u, v) = 1$, and (ii) those special non-self-adjoint equations (2.14) and (2.15) which have coefficients that satisfy (2.16).

3. The iterated Laplace transformation

In this section we shall briefly review the iterated application of the classical Laplace transformation to linear wave equations in $1+1$ dimensions. Our formulation will be based on earlier work by Kundt and Newman [10], but will include a notational innovation appropriate to the applications in this paper. We shall also emphasize two independent specializations of the basic equations that will be important to us later.

Given the generic wave equation (2.1) we put $j_0 \equiv k$, $j_1 \equiv j$, and $\phi_0 \equiv \phi$ so that (2.1) takes the form

$$(\partial_v j_0 \partial_u - j_1)\phi_0 = 0. \tag{3.1}$$

If we inductively define a doubly infinite sequence $\{j_k\}_{k\in\mathbb{Z}}$ by

$$j_{k+1}/j_k = j_k/j_{k-1} - \partial_u[(\partial_v j_k)/j_k] \qquad k \in \mathbb{Z} \tag{3.2}$$

and a second doubly infinite sequence $\{\phi_k\}_{k\in\mathbb{Z}}$ by the two first-order conditions

$$j_{k+1}\phi_{k+1} = j_k \partial_u \phi_k \qquad j_k\phi_{k-1} = \partial_v(j_k\phi_k) \qquad k \in \mathbb{Z} \tag{3.3}$$

then (3.3) alone imply

$$(\partial_v j_k \partial_u - j_{k+1})\phi_k = 0 \qquad k \in \mathbb{Z} \tag{3.4}$$

as the latter are obvious integrability conditions for the former, while it was shown in [10] and [11] that (3.4) imply (3.3) modulo (3.2). It can also be shown that each of the *v-normal form* equations in (3.4) has as its *u-normal form* equation

$$(\partial_u l_k \partial_v - l_{k-1})\psi_k = 0 \qquad k \in \mathbb{Z} \tag{3.5}$$

where

$$j_k l_k = 1 \qquad \phi_k = l_k \psi_k \tag{3.6}$$

and the new coefficient functions satisfy an analogue of (3.2):

$$l_{k-1}/l_k = l_k/l_{k+1} - \partial_v[(\partial_u l_k)/l_k] \qquad k \in \mathbb{Z}. \tag{3.7}$$

It should be emphasized that the v-normal form of the equation with coefficients j_k, j_{k+1} has as its u-normal form the equation with coefficients l_k, l_{k-1}. It is clear that any wave equation in (3.4) or (3.5) is equivalent to every equation in (3.4) and (3.5) in the sense that a solution to any one equation generates a solution to every equation by differentiation, through (3.6) and (3.3). It is the first and second of (3.3) that are, by definition, the Laplace transformations of (3.4) and (3.5), respectively.

Let us turn now to the ostensibly unrelated matter of TL dynamics. Toda's original work concerned a system with one independent variable, time t, and a set of dependent variables, $\{x_k\}_{k\in\mathbb{Z}}$, that can be interpreted as the absolute displacements of exponentially interacting lattice sites. If we take the standard generalization of Toda's original system where a second independent variable x has been added and we have transformed t and x to u and v according to

$$t \equiv v + u \qquad x \equiv v - u \tag{3.8}$$

and where we choose to change to new dependent variables $\{j_k\}_{k\in\mathbb{Z}}$ related to Toda's absolute displacements by

$$j_k \equiv \exp x_k \qquad k \in \mathbb{Z} \tag{3.9}$$

the equations defining TL dynamics are precisely (3.2), equations essential in defining the Laplace transformations of (3.4). We shall refer to this system as the *absolute two-dimensional TL* (ATTL) dynamical system. It immediately follows that there is a natural one-to-one and onto correspondence between the set of motions of the ATTL, represented by the set of sequences $\{j_k\}_{k\in\mathbb{Z}}$ that satisfy (3.2) and which we shall denote by $\mathcal{P}$, and the set of sequences of wave equations of the form (3.1) that are Laplace transformation equivalent, i.e. their coefficients also satisfy (3.2), and which we shall denote by $\mathcal{V}$. We represent this bijection by

$$\mathcal{T}_{\mathcal{V}} : \mathcal{V} \to \mathcal{P} \tag{3.10}$$

which has been applied in [6, 11–13] to obtain various useful results about Toda lattice motions and linear wave equations.

Our concern in what follows will be with sequences of wave equations containing a self-adjoint equation of the form (2.2), and with sequences that turn out to contain both the non-self-adjoint equation (2.14) and the v-normal form of the non-self-adjoint equation (2.15). Since $j(u, v) = 1$, it is natural to reflect the special role of this coefficient in the sequence by indexing it as j_0, in which case (2.2) becomes

$$(\partial_v \partial_u - j_1)\phi_0 = 0. \tag{3.11}$$

It follows from (3.1) and $j_0(u, v) = 1$ that

$$j_{-k} = 1/j_k \qquad k \in \mathbb{Z} \tag{3.12}$$

which emphasizes the central place of j_0 in the sequence. From the point of view of TL, motions (3.9) and (3.12) show that in terms of Toda's original variables a class of wave equations containing an explicitly self-adjoint equation corresponds under (3.10) to an anti-symmetrical TL motion with a fixed centre element (indexed by 0). On the other hand (2.14) has coefficients restricted by (2.16) which means that the sequence for this wave equation has a natural central *pair* of elements. In order that the indexing reflects this it is useful to depart from the usual practice and index the coefficients in this case with half-integers; this will prove desirable in other respects as well. Thus, we supplement equations (3.2)–(3.7) by a second, obviously equivalent, set of equations by replacing every index k by $k + \frac{1}{2}$. It will be sufficient for our purposes to explicitly display just the alternative form of (3.2):

$$\partial_u \left[(\partial_v j_{k+\frac{1}{2}})/j_{k+\frac{1}{2}}\right] = j_{k+\frac{1}{2}}/j_{k-\frac{1}{2}} - j_{k+\frac{3}{2}}/j_{k+\frac{1}{2}} \qquad k \in \mathbb{Z}. \tag{3.13}$$

In this notation (2.14) will become the equation

$$(\partial_v j_{-\frac{1}{2}} \partial_u - j_{\frac{1}{2}})\phi_{-\frac{1}{2}} \tag{3.14}$$

with condition (2.16) taking the form

$$\partial_v(1/j_{-\frac{1}{2}}) = \partial_u j_{\frac{1}{2}}. \tag{3.15}$$

In the particular case where $j_{\frac{1}{2}}(v,u) = j_{\frac{1}{2}}(v+u) = j_{\frac{1}{2}}(t)$, and similarly for $j_{-\frac{1}{2}}(u,v)$, (3.15) is satisfied by taking $j_{-\frac{1}{2}} = 1/j_{\frac{1}{2}}$ and it follows from (3.13) that

$$j_{-(k+\frac{1}{2})} = 1/j_{k+\frac{1}{2}} \tag{3.16}$$

mirroring the result (3.12) for self-adjoint equations. The central role of the pair of elements $j_{\frac{1}{2}}$, $j_{-\frac{1}{2}}$ holds even in the case where (3.16) does not. The TL motions that satisfy (3.16) are, in terms of Toda's original variables, antisymmetrical motions without a fixed central element, with $-\frac{1}{2}$ and $\frac{1}{2}$ indexing the central pair of elements.

Two independent specializations of (3.2) and (3.13) will be of interest in what follows, and we give the appropriate formulae here for future reference. On the one hand if we define

$$q_k \equiv j_{k+\frac{1}{2}}/j_{k-\frac{1}{2}} \qquad k \in \mathbb{Z} \tag{3.17}$$

then taking the difference of two instances of (3.13) with adjacent indices gives

$$\partial_u\left[(\partial_v q_k)/q_k\right] = -q_{k+1} + 2q_k - q_{k-1} \qquad k \in \mathbb{Z} \tag{3.18}$$

while if we define

$$q_{k+\frac{1}{2}} \equiv j_{k+1}/j_k \qquad k \in \mathbb{Z} \tag{3.19}$$

then doing the same with (3.2) results in

$$\partial_u\left[\partial_v q_{k+\frac{1}{2}}/q_{k+\frac{1}{2}}\right] = -q_{k+\frac{3}{2}} + 2q_{k+\frac{1}{2}} - q_{k-\frac{1}{2}} \qquad k \in \mathbb{Z}. \tag{3.20}$$

We shall refer to (3.18) and (3.20) as defining the *relative* two-dimensional TL (RTTL) dynamics since a ratio of j_k's corresponds to a difference of x_k's under (3.9). The mixing of integer and half-integer indices in the definition of the q_k's is appropriate as (3.12) implies that

$$q_{-(k+\frac{1}{2})} = q_{k+\frac{1}{2}} \qquad k \in \mathbb{Z} \tag{3.21}$$

while (3.16) implies that

$$q_{-k} = q_k \qquad k \in \mathbb{Z} \tag{3.22}$$

i.e. that a central element for the j_k's means a central pair for the $q_{k+\frac{1}{2}}$'s, and *vice versa*. It is clear that, in general, (3.18) is not equivalent to (3.13); however, in the special case where (3.16) holds they are equivalent, as a knowledge of the q_k's combined with (3.16) determines the $j_{k+\frac{1}{2}}$'s, and likewise (3.20) is equivalent to (3.2) if and only if (3.12) holds.

A different specialization follows if we assume that

$$j_k(u,v) = j_k(v+u) = j_k(t) \qquad j_{k+\frac{1}{2}}(u,v) = j_{k+\frac{1}{2}}(v+u) = j_{k+\frac{1}{2}}(t) \qquad k \in \mathbb{Z} \tag{3.23}$$

in which case (3.2) and (3.13) specialize to

$$(j_k'/j_k)' = j_k/j_{k-1} - j_{k+1}/j_k \qquad k \in \mathbb{Z} \tag{3.24}$$

and

$$(j'_{k+\frac{1}{2}}/j_{k+\frac{1}{2}})' = j_{k+\frac{1}{2}}/j_{k-\frac{1}{2}} - j_{k+\frac{3}{2}}/j_{k+\frac{1}{2}} \qquad k \in \mathbb{Z} \tag{3.25}$$

since $\partial_v = \partial_u = \mathrm{d}/\mathrm{d}t$, and where the prime means differentiation with respect to t. We shall call (3.24) and (3.25) absolute *one-dimensional* TL (AOTL) dynamics. If one does both specializations the result is

$$(q'_k/q_k)' = -q_{k+1} + 2q_k - q_{k-1} \qquad k \in \mathbb{Z} \tag{3.26}$$

and

$$(q'_{k+\frac{1}{2}}/q_{k+\frac{1}{2}})' = -q_{k+\frac{3}{2}} + 2q_{k+\frac{1}{2}} - q_{k-\frac{1}{2}} \qquad k \in \mathbb{Z} \tag{3.27}$$

which are the *relative one-dimensional* TL (ROTL) dynamical systems. The known connection between the TL and KvML dynamical systems to be discussed in the next section holds in the most special case corresponding to (3.26) and (3.27), and to either of the two generalizations (3.24) and (3.25), or (3.18) and (3.20), but not for the most general case (3.2) and (3.13). It is somewhat unexpected that the interpolation result for the relative one-dimensional lattices can be generalized to either the case of the absolute dynamics, or to the case of two-dimensional dynamics, but not to absolute two-dimensional dynamics. This is the anomaly in the relationship between the two types of lattice dynamics that was referred to in the introduction and it is even more surprising in the light of our results in section 5.

If we restrict the domain of (3.10) to be $\mathcal{V}_0$, the sequences of wave equations that include one that is self-adjoint, the range is restricted to ATTL motions for which (3.12) holds, and if we represent these anti-symmetrical motions about a fixed centre element by $\mathcal{P}_0$ we have the restricted bijection

$$\mathcal{T}_0 : \mathcal{V}_0 \to \mathcal{P}_0 \tag{3.28}$$

while if we restrict the domain of (3.10) to be the sequences of wave equations including one which has coefficients that satisfy (3.15), which we call $\mathcal{V}_{\frac{1}{2}}$, which restricts the range to the family of motions with a central pair of elements, and represent these motions by $\mathcal{P}_{\frac{1}{2}}$ we have a restriction of (3.10) to

$$\mathcal{T}_{\frac{1}{2}} : \mathcal{V}_{\frac{1}{2}} \to \mathcal{P}_{\frac{1}{2}}. \tag{3.29}$$

It is the two restricted bijections that we apply later.

4. Interpolating lattice motions

In this section we shall review the defining equations of the KvML and the connection between its motions and those of the TL. The system of first-order ordinary differential equations

$$z'_{k+\frac{1}{2}} = z_{k+\frac{1}{2}}(z_{k+1} - z_k) \qquad z'_k = z_k(z_{k-\frac{1}{2}} - z_{k+\frac{1}{2}}) \qquad k \in \mathbb{Z} \tag{4.1}$$

is a common representation of KvML dynamicals [2]. If we put

$$z_k \equiv j_k/j_{k-\frac{1}{2}} \qquad z_{k+\frac{1}{2}} \equiv j_{k+\frac{1}{2}}/j_k \qquad k \in \mathbb{Z} \tag{4.2}$$

then differentiating z_k, using both the equations in (4.1), and repeating the process yields (3.26), the integer indexed ROTL equations. If we follow the same route starting with $z_{k+\frac{1}{2}}$ the result is (3.27), the half-integer indexed ROTL equations. Thus, we shall refer to (4.1) as the *relative one-dimensional* KvML (ROKvML) dynamical system, to distinguish it from two related systems to be defined below, and have derived the result that each of the motions of the first-order ROKvML system is comprised of two interpolated motions of the second-order ROTL system. We now reverse, as far as possible, the process of specialization that we went through with the TL dynamics. The system of coupled first-order ordinary differential equations

$$j_k'/j_k = -j_{k+\frac{1}{2}}/j_k + j_k/j_{k-\frac{1}{2}} \qquad j_{k+\frac{1}{2}}'/j_{k+\frac{1}{2}} = j_{k+1}/j_{k+\frac{1}{2}} - j_{k+\frac{1}{2}}/j_k \qquad k \in \mathbb{Z} \tag{4.3}$$

which imply (4.1), also imply, as in the preceding calculation, that

$$(j_k'/j_k)' = j_k/j_{k-1} - j_{k+1}/j_k \qquad k \in \mathbb{Z} \tag{4.4}$$

and similarly that

$$(j_{k+\frac{1}{2}}'/j_{k+\frac{1}{2}})' = j_{k+\frac{1}{2}}/j_{k-\frac{1}{2}} - j_{k+\frac{3}{2}}/j_{k+\frac{1}{2}} \qquad k \in \mathbb{Z} \tag{4.5}$$

which are the AOTL equations. Hence, (4.3) will be referred to as the *absolute* one-dimensional KvML (AOKvML) dynamical system, and the interpolation result just obtained for the relative one-dimensional systems has generalized to the absolute one-dimensional case. If we postulate instead that z_k depend on both u and v and satisfy

$$\partial_u z_{k+\frac{1}{2}} = z_{k+\frac{1}{2}}(z_{k+1} - z_k) \qquad \partial_v z_k = z_k(z_{k-\frac{1}{2}} - z_{k+\frac{1}{2}}) \qquad k \in \mathbb{Z} \tag{4.6}$$

(4.6) is obviously a generalization of (4.1), and if we write

$$q_{k+\frac{1}{2}} = z_{k+1} z_{k+\frac{1}{2}} \qquad q_k = z_{k+\frac{1}{2}} z_k \qquad k \in \mathbb{Z} \tag{4.7}$$

which follow from (3.17), (3.19) and (4.2), it follows that

$$\begin{aligned} \partial_u(\partial_v q_k/q_k) &= \partial_u(\partial_v z_{k+1}/z_{k+1}) + \partial_v(\partial_u z_{k+\frac{1}{2}}/z_{k+\frac{1}{2}}) \\ &= -\partial_u(z_{k+\frac{1}{2}} - z_{k-\frac{1}{2}}) + \partial_v(z_{k+1} - z_k) \qquad k \in \mathbb{Z} \\ &= -q_{k+1} + 2q_k - q_{k-1} \end{aligned} \tag{4.8}$$

where four terms have cancelled in pairs, while similarly

$$\partial_u(\partial_v q_{k+\frac{1}{2}}/q_{k+\frac{1}{2}}) = -q_{k+\frac{3}{2}} + 2q_{k+\frac{1}{2}} - q_{k-\frac{1}{2}} \qquad k \in \mathbb{Z}. \tag{4.9}$$

But (4.8) is precisely (3.18), while (4.9) is precisely (3.20), so we can view the set $\{q_k\}_{k\in\mathbb{Z}}$ as being constucted out of two interpolated motions of the half-integer indexed RTTL, and similarly for the set $\{q_{k+\frac{1}{2}}\}_{k\in\mathbb{Z}}$, with reference to the integer indexed RTTL. Since it is the *relative* TL equations that arise we shall designate (4.6) to be the relative *two-dimensional* KvML (RTKvML) dynamical equations. This system has been studied before, for example in [3].

One naturally expects that to complete the picture there should exist a first-order system of coupled partial differential equations to be called the absolute two-dimensional KvML dynamical system, which has motions that comprise two interpolated motions of the second-order ATTL dynamics, but we are unaware of any set of equations that play this role. In particular, if we replace the primes in (4.3) by a u derivative in one equation and a v derivative in the other the calculation fails to generalize in the required way.

5. Wave splitting and the KvML

We return to the wave splitting formulae derived in section 2 and begin with the self-adjoint wave equation that we are solving in v-normal form:

$$(\partial_v j_0 \partial_u - j_1)\phi_0 = 0 \tag{5.1}$$

with

$$j_0 = 1 \qquad j_1 = j \qquad \phi_0 = \phi. \tag{5.2}$$

Turning to the pair of non-self-adjoint equations satisfied by the one-way component waves, we first take equation (2.15), which is satisfied by both ϕ^{Aod} and ϕ^{Rev}, and identify it with the u-normal form equation

$$(\partial_u l_{\frac{1}{2}} \partial_v - l_{-\frac{1}{2}})\psi_{\frac{1}{2}} = 0 \tag{5.3}$$

by putting

$$1/b = l_{\frac{1}{2}} \qquad f = l_{-\frac{1}{2}} \qquad \psi_{\frac{1}{2}} = \psi. \tag{5.4}$$

The v-normal form of (5.3) is

$$(\partial_v j_{\frac{1}{2}} \partial_u - j_{\frac{3}{2}})\phi_{\frac{1}{2}} \tag{5.5}$$

as was discussed in section 3, and one application of the Laplace transformation, defined by (3.2) and (3.7), to (5.5) yields the equation

$$(\partial_v j_{-\frac{1}{2}} \partial_u - j_{\frac{1}{2}})\phi_{-\frac{1}{2}} = 0. \tag{5.6}$$

But according to (3.6) and (5.4)

$$j_{-\frac{1}{2}} = 1/l_{-\frac{1}{2}} = 1/f \qquad j_{\frac{1}{2}} = 1/l_{\frac{1}{2}} = b \tag{5.7}$$

so (5.6) becomes (2.14), the non-self-adjoint equation for ϕ^{Aev} and ϕ^{Rod}. Thus we have established that the two pairs of one-way component waves satisfy a pair of non-self-adjoint wave equations that are equivalent under a Laplace transformation (and a change of normal form). There was no *a priori* reason to expect that these different equations should be so simply related, and this result endows the wave splitting introduced in section 2 with a certain formal attractiveness. Since the Laplace transformation was seen in section 4 to be intimately related to TL dynamics, it is natural to look for some further connection between the wave splitting and lattice dynamics and the interpolation result of section 4 is an obvious candidate.

If we now interpolate the coefficients of the equivalence class of equations containing (5.6), which include (5.7), with the coefficients of the equivalence class of equations containing (5.1), which include (5.2), we obtain the sequence $\{j_k\}_{2k\in\mathbb{Z}}$ which has the five central elements

$$\ldots, 1/j, 1/f, 1, b, j, \ldots. \tag{5.8}$$

with the rest of the sequence generated by these elements. We are interested in (5.8) as a possible KvML motion, as it is an interpolation of two TL motions, but the latter are ATTL motions and we have no candidate for ATKvML dynamics. However, if we take the ratios of successive elements of (5.8) to form the sequence

$$\ldots, j/f, f/1, b/1, j/b, \ldots. \tag{5.9}$$

and identify this with the sequence

$$\ldots, z_{-\frac{1}{2}}, z_0, z_{\frac{1}{2}}, z_1, \ldots \tag{5.10}$$

then the two equations of (4.6), the RTKvML, indexed by $k = 0$

$$\partial_v z_0 = z_0(z_{-\frac{1}{2}} - z_{\frac{1}{2}}) \qquad \partial_u z_{\frac{1}{2}} = z_{\frac{1}{2}}(z_1 - z_0) \tag{5.11}$$

become

$$\partial_v f = f(j/f - b/1) \qquad \partial_u b = b(j/b - f/1) \tag{5.12}$$

which are precisely our basic equations (2.11). Each additional equation in (4.6) indexed by a value of k different from zero includes a new z_k, so each additional equation is automatically satisfied. Thus, we see that under the maps $\mathcal{T}_0$ and $\mathcal{T}_{\frac{1}{2}}$ the classes of equations containing (5.1) and (5.5) map, respectively, to elements of $\mathcal{Q}_0$ and $\mathcal{Q}_{\frac{1}{2}}$ that are the interpolants of a RTKvML motion.

If instead we assume that the coefficient j in the wave equation (5.1) satisfies $j(u, v) = j(t)$, and assume that $b(u, v) = f(u, v) = g(t)$, the basic equations for the splitting functions, (2.11), become the same equation, i.e.

$$g'/g + g = j(t)/g \tag{5.14}$$

and the sequence (5.8) specializes to

$$\ldots, 1/j, 1/g, 1, g, j, \ldots. \tag{5.15}$$

If we identify (5.15) with

$$\ldots, j_{-1}, j_{-\frac{1}{2}}, 1, j_{\frac{1}{2}}, j_1, \ldots \tag{5.16}$$

it is easy to confirm that the $k = 0$ instances of (4.3) are satisfied. Thus, the sequence (5.15) is an AOKvML motion as all the other equations in that system are identically satisfied by defining the remaining elements of the sequence. Of course, the double specialization to a strictly t dependent sequence (5.9) leads to RTKvML motions.

Naturally, one would like to show that the original sequence (5.8), where the elements depend on u and v in a more general way, is a solution of a system that would be called the absolute two-dimensional KvML system; however, the sequence (5.8) does not satisfy any pair of coupled first-order partial differential equations that could be put forward as a candidate for the role, nor, as was mentioned above, is there any possibility suggested by lattice dynamics.

6. Conclusion

The wave-decomposition results derived in section 2 have something in common with equations given by Bruckstein and Kailath [7]. If we specialize (2.2) by assuming that $j(u, v) = j(v+u) = j(t)$, then we can satisfy (2.11) by assuming that $b = f = g(t)$, with g a solution of (5.14). This equation and its role in splitting solutions of (2.2) in the special case where $j(u, v)$ depends exclusively on t are discussed at the beginning of their paper, however, none of the series (2.3), (2.4) and (2.6), nor the non-self-adjoint equations (2.14) and (2.15) appear in their work.

A thoroughly investigated class of frequency domain wave splittings for the wave equation (2.2) with $j(u, v) = j(x)$ originated with Bremmer [8, 9], and can be compared with the splittings introduced here. The Bremmer-type series solutions, and splittings, result from the approximation of the inhomogeneous medium through which the wave is travelling by layers within each of which the wave equation can be exactly solved. After taking into account all transmissions and reflections at the interfaces between the approximating layers, the layer thicknesses are allowed to go to zero, and interesting infinite series solutions and local wave splittings are obtained. Despite the similarity in the two approaches the series which result from that layering approach are not the same as those we obtain by building in continuous reflection from the start. To begin with there are an infinite variety of Bremmer-type splittings for each wave equation because there are an infinite variety of media in which the wave equation has a general solution that propagates without scattering [6]; this variety of splittings is in no way constrained by the particular wave equation one is trying to solve. There are also an infinite variety of splittings for each wave equation with our approach; however, they are indexed by the variety of particular solutions of that equation and so are peculiar to it. Of course, the non-uniqueness of both classes of splittings argues strongly against the possibility that in either case local physical significance can be attributed to the decomposition of wave phenomena into two one-way flows. More significantly, from our point of view, the one-way waves resulting from the Bremmer approach do not seem to satisfy any simple wave equations, precluding any result for those component waves comparable to that derived for ours in section 5.

There are open questions prompted by the formal result obtained in section 5. In [11] the connection between linear wave equations and TL motions described in section 3 was generalized to one between systems of linear wave equations and non-Abelian TL motions. The results of the last section may extend to this wider context. It is also puzzling that while the wave decomposition results of section 2 apply to arbitrary self-adjoint wave equations, i.e. to arbitrary sequences of j_k's with $j_{-k} = 1/j_k$, only after either assuming that the the j_k's are strictly t-dependent, or after going to the sequence of q_k's, i.e. to relative displacements in lattice terms, can we relate the wave decompositions to KvML lattice dynamics. There is a natural role in the scheme of things for a two-dimensional KvML dynamics of absolute displacements, but no candidate to fill it.

Acknowledgment

This work has been supported in part by the Natural Sciences and Engineering Research Council of Canada

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